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CAPS AND VERONESE VARIETIES IN PROJECTIVE GALOIS SPACES*

Francesco MAZZOCCA and Nicola MELONE

Istituto di Matematica "R. Caccioppoli", Università di Napoli, Via Mezzocannone 8, 80134 Napoli, Italy

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In this paper a projective combinatorial characterization of Veronese varieties in a Galois space $PG(n, q)$ is given.

1. Introduction

Let $V(r, F)$ denote the Veronese variety, i.e. the projective model of the complete linear system of quadrics in an r-dimensional projective space $PG(r, F)$ over a commutative field F. $V(r, F)$ is an r-dimensional irreducible variety embedded in a projective space of dimension $\frac{1}{2}r(r+3)$ and the family of its conics determines on $V(r, F)$ a structure of projective space.

The geometry on Veronese varieties has been studied both within classical algebraic and Galois geometry. Indeed, remarkable characteristic properties and nice projective generations of these varieties are known (see the references and their respective bibliographies). In particular, an intrinsic characterization exists of the Veronese surface $V(2, \mathbb{C})$ over the complex field based on properties of its conics and under the further assumption that the embedding space has dimension five [1].

It is easy to check that the following properties hold for a Veronese variety $V(r, F)$:

(i) *Any two distinct points on $V(r, F)$ belong to a conic in $V(r, F)$.*

(ii) *Any two planes intersecting $V(r, F)$ in conics meet on $V(r, F)$, or they are disjoint.*

(iii) *If c is a conic in $V(r, F)$ and p a point in $V(r, F) - c$, the tangents at p to the conics in $V(r, F)$ which are incident with c and pass through p are coplanar.*

The above properties suggested the investigation of sets C in an n-dimensional projective space $PG(n, q)$ over a Galois field $GF(q)$, containing no lines and a family S of conics satisfying properties (i), (ii) and (iii). Under these assumptions, an integer $r > 0$ is proved to exist such that $n = \frac{1}{2}r(r+3)$, C is the Veronese variety

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$V(r, q)$, i.e. the projective model of the complete linear system of quadrics in $PG(r, q)$, and S is the family of all conics in $V(r, q)$ (Section 4, Theorem 2).

2. Veronese caps

In the sequel, C and $\mathscr{C}(C)$ will denote a point set in $PG(n, q)$, containing no lines and spanning $PG(n, q)$, and the family of plane sections of C which are conics, respectively.

Let S be a subset of $\mathscr{C}(C)$ and assume $S \neq \emptyset$. Call *S-conics* the elements in S and *S-planes* the planes in $PG(n, q)$ meeting C at S-conics. Denote by $\Sigma(S)$ the family of all S-planes. The set S will be said to be *linear* if (C, S) is a line space, i.e. any two distinct points in C belong to a unique S-conic. A line t in $PG(n, q)$ will be called an *S-tangent* to C at the point $p \in C$ if it is tangent to an S-conic c through p at p. If this occurs, we set $t = t(p, c)$. If S is linear, the unique S-conic through two distinct points p, q in C will be denoted by $[p, q]_S$, for short $[p, q]$ if no confusion arises. On the other hand, the notation (p, q) will be used for the line in $PG(n, q)$ through the points p and q.

In the sequel it will be assumed that $\mathscr{C}(C)$ is not empty and S satisfies:

(v_1) *Any two distinct points in C belong to an S-plane.*

(v_2) *Any two distinct S-planes meet on C, or they are disjoint.*

Notice that two distinct points in C belong to a unique S-conic, otherwise by (v_2) the line common to the S-planes through the two points would be contained in C, contradicting the assumptions on the set C. Thus, from (v_1) it follows that S is linear. Moreover, if a line t meets C at three distinct points, then by (v_1) and (v_2), $t \subset C$, a contradiction.

Consequently, the following result is true.

Proposition 2.1. *If S satisfies properties* (v_1) *and* (v_2), *then (C, S) is a line space. Moreover, any three distinct points in C are non-collinear, i.e. C is a cap.*

Next, if two distinct S-conics c and c' have the same tangent t at a common point p, the planes containing c and c' meet at t. Thus, by (v_2), t belongs to C, contradicting the assumptions on C. Therefore, the following holds.

Proposition 2.2. *An S-tangent to C at a point p is tangent to a unique S-conic through p.*

(C, S) will be called a *Veronese cap* if it satisfies (v_1), (v_2) and

(v_3) *If c is an S-sonic and p a point in $C - c$, the S-tangents at p to the S-conics through p which are incident with c are coplanar.*

From now on, an S-conic c and a point p in $C - c$ being given, $\pi(p, c)$ will denote the plane in $PG(n, q)$ containing the S-tangents at p to the S-conics $[p, x]$ where $x \in c$.

Remark 1. Let S be the family of all conics belonging to a Veronese variety $V(r, F)$ in $\mathrm{PG}(\frac{1}{2}r(r+3), F)$, F a commutative field. It is straightforward that $(V(r, F), S)$ is a Veronese cap.

Proposition 2.3. *The line space* (C, S) *is a projective space.*

Proof. It is enough to prove that three non-collinear points p, p', p'' in the line space (C, S) belong to a projective plane contained in (C, S). By Proposition 2.2, the number of S-tangents at p to the S-conics $[p, x]$, with $x \in [p', p'']$, equals the number $q+1$ of points on the S-conic $[p', p'']$. Thus, these lines form a complete pencil. Next, let a, b be two points other than p in $[p, p']$ and $[p, p'']$, respectively; the S-conic $[a, b]$ meets each S-conic $[p, x]$ with $x \in [p', p'']$. Indeed, by Proposition 2.2, the S-tangents to $[p, p']$ and $[p, p'']$ at p are distinct and by (v_3) they belong to the intersection of $\pi(p, [p', p''])$ and $\pi(p, [a, b])$. Thus, $\pi(p, [p', p'']) = \pi(p, [a, b]) = \pi$. If a point x existed in $[p', p'']$ such that $[p, x] \cap [a, b] = \emptyset$, then there would be at least $q+2$ lines through p in π, a contradiction.

Therefore, the set $V = \bigcup \{[p, x] : x \in [p', p'']\}$ is a subspace in (C, S), since the S-conic through two distinct points in V belongs to V. Moreover, both the number of points and of S-conics in V is precisely q^2+q+1; consequently, V is a projective plane of order q contained in C.

The *index* of a Veronese cap (C, S) is defined as the dimension of the projective space (C, S). A Veronese cap (C, S) of index r together with its projective structure will be denoted by C_r.

For any point p in C_r, set

$$T(p) = \bigcup \{t(p, c) : c \in S \text{ and } p \in c\}.$$

Proposition 2.4. *For any point p in C_r, $T(p)$ is an r-subspace in* $\mathrm{PG}(n, q)$.

Proof. Let x, y be any two distinct points in $T(p)$ and suppose x, y are not collinear with p, otherwise $(x, y) \subseteq T(p)$, trivially. The lines (p, x) and (p, y) are distinct and tangent to two distinct S-conics c and c'. Given $a \in c - \{p\}$ and $b \in c' - \{p\}$, the plane $\pi(p, [a, b])$ coincides with that spanned by x, y and p. Thus, $(x, y) \subseteq T(p)$, since $(x, y) \subseteq \pi(p, [a, b])$ and $\pi(p, [a, b]) \subseteq T(p)$; hence, $T(p)$ is a subspace in $\mathrm{PG}(n, q)$. Moreover, associating with every S-conic through p its S-tangent at p, a collineation arises between the star of lines in C_r through p and the star of lines in $T(p)$ through p; therefore, $T(p)$ has dimension r.

For any point $p \in C_r$, the r-subspace $T(p)$ will be called the *tangent space* to the Veronese cap C_r at p.

Proposition 2.5. *Let c and c' be two S-conics in C_r. If there exist two distinct points $a \in c$ and $b \in c'$ such that $t(a, c) \cap t(b, c') \neq \emptyset$, then $c = c'$.*

Proof. If $t(a, c) = t(b, c')$, the statement follows from Proposition 2.1. Assume $t(a, c) \neq t(b, c')$ and let p be the point $t(a, c) \cap t(b, c')$. If $c \neq c'$, the planes π and π', containing c and c' respectively, are distinct and meet at p, so that by (v_2), $p \in C_r$ and $p = c \cap c'$. If $p = a$, $t(b, c')$ intersects c' in the distinct points p and b, contradicting the assumption that $t(b, c')$ is tangent to c' at b. Thus, $p \neq a$ and, by the same argument, $p \neq b$. Therefore, $t(a, c)$ and $t(b, c')$ cannot be S-tangent to c and c' at p, a contradiction.

Proposition 2.6. *Two distinct tangent spaces to C_r have a unique common point and this point does not belong to C_r.*

Proof. Let p, q be two distinct points in C_r. The lines $t(p, [p, q])$ and $t(q, [p, q])$ meet at a point a and obviously $a \in (T(p) \cap T(q)) - C_r$. Now, given a point $x \in T(p) \cap T(q)$, the lines (p, x) and (q, x) are S-tangents to C_r at p and q, respectively. Therefore, by Propositions 2.1 and 2.5, (p, x) and (q, x) are the tangents to the S-conic $[p, q]$ at p and q, whence $x = a$.

Proposition 2.7. *If $p \in C_r$ and $x \in T(p) - \{p\}$, then there exists a unique point $q \in C_r - \{p\}$ such that $x \in T(q)$.*

Proof. Let x be a point in $T(p)$ with $x \neq p$. The line (p, x) is tangent to an S-conic c at p. Thus, x belongs to the plane π containing c and through x another tangent t passes to c other than (p, x). If q is the point at which t is tangent to c, then $x \in T(q)$. From Proposition 2.5 it follows that $T(p)$ and $T(q)$ are the only tangent spaces through x.

3. Veronese caps of index two

In this section the Veronese caps of index two will be characterized. For this purpose, the following propositions are needed.

Proposition 3.1. *Let p be a point in C_2. Any S-plane not through p is skew with the tangent plane $T(p)$ to C_2.*

Proof. Let π be an S-plane not through p and suppose that π meets $T(p)$ at a point x. Then, $x \neq p$ and the line (p, x) is tangent to an S-conic c in C_2 at p. The plane containing c meets the plane π at x. Thus, by (v_2), $x \in C_2$ i.e. $x = p$. Hence, $p \in \pi$, a contradiction.

The next result is a straightforward consequence of Proposition 3.1.

Proposition 3.2. *If C_2 is a Veronese cap of index two in* PG(n, q), *then* $n = 5$.

Proposition 3.3. *Let $B = \{p_1, p_2, p_3\}$ be a basis of the projective plane C_2 and set $p_{ij} = T(p_i) \cap T(p_j)$, $i, j = 1, 2, 3$, $i \neq j$. Then the set $\tilde{B} = \{p_1, p_2, p_3, p_{12}, p_{13}, p_{23}\}$ is a basis of* PG$(5, q)$.

Proof. First of all, remark that the points in B do not belong to the same S-conic; therefore, the sets $\{p_1, p_{12}, p_{13}\}$, $\{p_2, p_{12}, p_{23}\}$ and $\{p_3, p_{13}, p_{23}\}$ are bases in $T(p_1)$, $T(p_2)$ and $T(p_3)$, respectively. The points p_1, p_2, p_{12}, p_{13}, p_{23} are independent, since $\{p_1, p_{12}, p_{13}\}$ spans the plane $T(p_1)$ and $T(p_1)$ and the line (p_2, p_{23}) are skew. If p_3 belongs to the hyperplane s in PG$(5, q)$ spanned by $\tilde{B} - \{p_3\}$, then $T(p_3) \subseteq s$, since s and $T(p_3)$ share three independent points: p_3, p_{13}, p_{23}. The S-plane π through p_1, p_2 does not contain p_3; moreover, $T(p_3)$ and π have at least one common point since $\pi \subseteq s$ and $T(p_3) \subseteq s$, contradicting Proposition 3.1.

Proposition 3.4. *The projection of C_2 from a tangent plane $T(p)$ to C_2 is a quadric cone with vertex the plane $T(p)$.*

Proof. Let c be an S-conic not through a fixed point p on C_2. Given two distinct points x, y on $C_2 - \{p\}$, the subspaces spanned by $T(p) \cup \{x\}$ and $T(p) \cup \{y\}$ coincide iff the S-plane through x and y passes through p. Then, the projection of C_2 from $T(p)$ equals the projection of c from $T(p)$.

Proposition 3.5. *Any plane in* PG$(5, q)$ *not in $\Sigma(S)$ meets C_2 in at most three distinct points.*

Proof. Let π be a plane in PG$(5, q)$ meeting C_2 at the distinct points a, b, c, d. If the S-planes π_1 and π_2 through the S-conics $[a, b]$ and $[c, d]$ are distinct, they meet π at the lines (a, b) and (c, d), respectively. Then, the point $p = [a, b] \cap [c, d]$ belongs to π, since it coincides with the point $\pi_1 \cap \pi_2 = (a, b) \cap (c, d)$. Thus, at least one of the conics $[a, b]$ and $[c, d]$ has three collinear points, a contradiction. Therefore, $\pi_1 = \pi_2 = \pi$.

From Proposition 3.5 easily follows the next one.

Proposition 3.6. *Let p, q be two distinct points on C_2 and π the S-plane through p and q. The planes in* PG$(5, q)$ *through the line (p, q), other than π and meeting C_2, are q^2 and each of them meets C_2 at a unique point.*

Proposition 3.7. *Let p, q two distinct points on C_2 and $s(p, q)$ a 3-subspace in* PG$(5, q)$ *skew with the line (p, q). Then the projection of C_2 from (p, q) on $s(p, q)$ is a hyperbolic quadric cone with vertex the line (p, q).*

Proof. Let π be the S-plane through p and q. By Proposition 3.5 the projection $\bar{C}$ of $C_2-[p,q]$ from (p,q) on $s(p,q)$ has q^2 points. Moreover, $[p,q]-\{p,q\}$ projects onto the point $x_0=s(p,q)\cap\pi$ and the points p, q blow up onto the lines $\bar{p}$ and $\bar{q}$ in $s(p,q)$ projections of the S-tangents at p and q, respectively. These lines $\bar{p}$ and $\bar{q}$ contains x_0, whence $\bar{C}$ has $(q+1)^2$ points. The S-conics c through p, q, other than $[p,q]$, projects onto lines $\bar{c}$ in $\bar{C}$ meeting at a point $\bar{p}$ and $\bar{q}$, respectively. These lines together with $\bar{p}$ and $\bar{q}$ split in the families $F=\{\bar{c}:c\in S-[p,q]$ and $p\in c\}\cup\{\bar{q}\}$ and $F'=\{\bar{c}:c\in S-[p,q]$ and $q\in c\}\cup\{\bar{p}\}$. Any point in $\bar{C}$ belongs to a unique line in F and in F'. Moreover, any two lines in the same family are skew and any two lines in different families are incident. Thus, $\bar{C}$ is a ruled quadric in $s(p,q)$ and the projection of C_2 from (p,q) equals the projection of the ruled quadric $\bar{C}$ from (p,q).

Propositions 3.3, 3.4 and 3.7 allow us to use the same argument in [8, n.6]. Thus, the next theorem holds.

Theorem 1. *Let C_2 be a Veronese cap of index two in* PG(n,q). *Then $n=5$ and C_2 is a Veronese surface.*

Remark 2. Let C_r be a Veronese cap and p a point in C_r. For any point x in $T(p)-\{p\}$, denote by x_p the unique point in $C_r-\{p\}$ such that $x\in T(x_p)$ (see Proposition 2.7).

Moreover, let $\omega_{r,p}:T(p)\to C_r$ be the mapping

$$\omega_{r,p}(x)=\begin{cases} p, & \text{if } x=p,\\ x_p, & \text{if } x\neq p.\end{cases} \tag{3.1}$$

When C_r is the Veronese variety $V(r,F)$ it is well known that $\omega_{r,p}$ is a collineation between $T(p)$ and the projective space C_r.

4. Veronese caps of index greater than two

Firstly notice that from Proposition 3.2 and Theorem 1 the next result follows.

Proposition 4.1. *A plane C_2 in the projective space C_r is a Veronese surface. Therefore, the subspace spanned by C_2 in* PG(n,q) *has dimension five.*

Let $\omega_{r,p}:T(p)\to C_r$ be the mapping defined by (3.1).

Proposition 4.2. *The mapping $\omega_{r,p}$ is a collineation between the tangent space $T(p)$ and the projective space C_r.*

Proof. By definition, $\omega_{r,p}$ is a bijection mapping lines in $T(p)$ through p onto

S-conics in C_r through p. Next, let t be a line in $T(p)$ not through p and π the plane spanned by p and t. By (v_3) and Proposition 2.2, the S-conics in C_r tangent at p to all lines in π through p form a C_2 in C_r. Moreover, π is the tangent plane to C_2 at p. Thus, from Proposition 4.1 and Remark 2 it follows that $\omega_{r,p}|_\pi$ is a collineation between π and C_2, whence $\omega_{r,p}(t)$ is an S-conic and the statement follows.

Given two distinct points p, q in C_r, by Proposition 2.7, any point $x \neq p$ in $T(p) - T(q)$ belongs to a tangent space to C_r, other than $T(p)$. Denote by x_{pq} the intersection between such space and $T(q)$ and consider the mapping $\omega_{pq} : T(p) \to T(q)$

$$\omega_{pq}(x) = \begin{cases} T(p) \cap T(q), & \text{if } x = p, \\ q, & \text{if } x = T(p) \cap T(q), \\ x_{pq}, & \text{if } x \neq p, q. \end{cases} \tag{4.1}$$

Clearly $\omega_{pq} = \omega_{r,q}^{-1} \circ \omega_{r,p}$ and the next proposition follows.

Proposition 4.3. *The mapping ω_{pq} is a collineation between the tangent spaces $T(p)$ and $T(q)$.*

Next, let $B = \{p_1, p_2, \ldots, p_{r+1}\}$ be a basis of the projective space C_r and set $p_{ij} = p_{ji} = T(p_i) \cap T(p_j)$, $i, j = 1, 2, \ldots, r+1$, $i \neq j$. By Propositions 4.2 and 4.3, for every $i = 1, 2, \ldots, r+1$ the set

$$B(T(p_i)) = \{p_{i1}, \ldots, p_{ii-1}, p_i, p_{ii+1}, \ldots, p_{ir+1}\} \tag{4.2}$$

is a basis of the tangent space $T(p_i)$. Furthermore, from Proposition 4.2 follows

Proposition 4.4. *For any tangent space T to C_r at a point $p \in C_r - B$, the set $B(T) = \{T \cap T(p_1), \ldots, T \cap T(p_{r+1})\}$ is a basis of T.*

The bases $B(T(p_i))$ and $B(T)$ of $T(p_i)$ and T, respectively, will be called bases *associated* with the basis B of C_r and in the sequel we set

$$\tilde{B} = B(T(p_1)) \cup B(T(p_2)) \cup \cdots \cup B(T(p_{r+1})). \tag{4.3}$$

By Proposition 2.6,

$$|\tilde{B}| = \tfrac{1}{2}r(r+3) + 1. \tag{4.4}$$

By Proposition 4.4, the subspace s in PG(n, q) spanned by $\tilde{B}$ contains all the tangent spaces to C_r, whence it contains C_r. Therefore, s coincides with PG(n, q) since C_r spans PG(n, q), and the next statement is proved.

Proposition 4.5. *For any basis B of the projective space C_r, the set $\tilde{B}$ spans* PG(n, q). *Consequently, $n \leq \frac{1}{2}r(r+3)$.*

Given a point p in C_r, let $s(p)$ be a subspace in $\mathrm{PG}(n, q)$ of dimension $n-r-1$ skew with the tangent space $T(p)$ and let f_p be the projection from $T(p)$ on $s(p)$. Set $C_p = f_p(C_r)$.

Proposition 4.6. *For any point p in C_r, the set C_p is a Veronese cap of index $r-1$ in $s(p)$.*

Proof. First of all note that if p is a point in C_r and π is an S-plane not through p, then

$$T(p) \cap \pi = \emptyset. \tag{4.5}$$

Indeed, if q is a point in $T(p) \cap \pi$ and π' is the plane containing the S-conic tangent to the line (p, q) at p, then $q \in \pi \cap \pi'$, so that, by (v_2), $q \in C_r$, i.e. $q = p$, a contradiction since $p \notin \pi$. From (4.5) it follows that, for any two distinct points $x, y \in C_r$, $f_p(x) = f_p(y)$ iff p belongs to the S-conic $[x, y]$. Therefore, given a C_{r-1} in C_r not through p, $f_{p|C_{r-1}}$ is a bijection between C_{r-1} and C_p. Moreover, f_p maps an S-plane π not through p onto a plane $\bar{\pi}$ in $s(p)$ and the set $\bar{c} = \bar{\pi} \cap C_p = f_p(\pi \cap C_r)$ is a conic on C_p, since it is the section between $\bar{\pi}$ and the quadric cone projecting the conic $c = \pi \cap C_{r-1}$ from $T(p)$. Moreover, $\bar{c}$ is non-degenerate since $\bar{\pi} \cap T(p) = \emptyset$. Set

$$\bar{\Sigma} = \{f_p(\pi) : \pi \in \Sigma(S) \text{ and } p \notin \pi\}, \qquad \bar{S} = \{\bar{\pi} \cap C_p : \bar{\pi} \in \bar{\Sigma}\}.$$

$(C_p, \bar{S})$ is a Veronese cap of index $r-1$ in $s(p)$. To prove this claim, let C_{r-1} be a fixed hyperplane in C_r not through p. Assume $\bar{x}, \bar{y}, \bar{z}$ are three collinear points in C_p and let x, y, z be the points in C_{r-1} such that $f_p(x) = \bar{x}$, $f_p(y) = \bar{y}$, $f_p(z) = \bar{z}$. The S-plane π through x and y does not contain z; otherwise, x, y, z are contained in an S-conic c not through p, so that $\bar{x}, \bar{y}, \bar{z}$ cannot be collinear since they belong to the conic $\bar{c} = f_p(c)$ in C_p. Moreover, the plane $\bar{\pi} = f_p(\pi)$ meets C_p at a conic $\bar{c}'$ through $\bar{x}, \bar{y}$ and $\bar{z} \notin \bar{\pi}$, since $z \notin \pi$. On the other hand, since $\bar{z}, \bar{y}, \bar{z}$ are collinear and $\bar{x}, \bar{y} \in \bar{\pi}$, $\bar{z} \in \bar{\pi}$, a contradiction. Thus, C_p is a cap in $s(p)$ and it is straightforward properties (v_1) and (v_2) hold for C_p. Finally, we prove that (v_3) holds for C_p. Let $\bar{a}$ be a point in C_p, $\bar{c}$ an $\bar{S}$-conic not through $\bar{a}$ and $\bar{x}, \bar{y}, \bar{z}$ three distinct points on $\bar{c}$. Moreover, let a, x, y, z and c be the points and the S-conic in C_{r-1} that f_p projects onto $\bar{a}, \bar{x}, \bar{y}, \bar{z}$ and $\bar{c}$, respectively. The tangents t_x, t_y, t_z at a to the S-conics $[a, x]$, $[a, y]$, $[a, z]$ are projected onto the tangents $t_{\bar{x}}, t_{\bar{y}}, t_{\bar{z}}$ at $\bar{a}$ to the $\bar{S}$-conics $[\bar{a}, \bar{x}]$, $[\bar{a}, \bar{y}]$, $[\bar{a}, \bar{z}]$. Hence, the lines $t_{\bar{x}}, t_{\bar{y}}, t_{\bar{z}}$ belong to the intersection between $s(p)$ and the subspace spanned by $T(p) \cup \pi(a, c)$. Furthermore, $T(p) \cap \pi(a, c) = \emptyset$. Otherwise, a point b in $T(p) \cap \pi(a, c)$ belongs to the S-tangent (p, b) at p to an S-conic through p and to the S-tangent (b, a) at a to an S-conic contained in C_{r-1}, contradicting Proposition 2.5. Thus, the subspace spanned by $T(p) \cup \pi(a, c)$ has dimension $r+3$, so that it meets $s(p)$ at a plane containing $t_{\bar{x}}, t_{\bar{y}}$ and $t_{\bar{z}}$. Therefore, property (v_3) holds and the proof is complete.

From Proposition 4.6 the next one follows.

Proposition 4.7. *If C_r is a Veronese cap in* $\mathrm{PG}(n, q)$, *then* $n = \frac{1}{2}r(r+3)$.

Proof. By Proposition 4.5, it suffices to prove that $n \geqslant \frac{1}{2}r(r+3)$. By Proposition 3.2, the statement is true for $r = 2$. Assume $r > 2$ and the statement true for $r-1$. By the induction hypothesis and Proposition 4.6, $n - r - 1 \geqslant \frac{1}{2}(r-1)(r+2)$, i.e. $n \geqslant \frac{1}{2}r(r+3)$.

Taking into account Proposition 4.7, Proposition 4.5 can be formulated as follows

Proposition 4.8. *For any basis B of the projective space C_r, the set $\tilde{B}$ is a basis of* $\mathrm{PG}(\frac{1}{2}r(r+3), q)$, *called the basis associated with B.*

Next, given the bases $B = \{p_1, \ldots, p_{r+1}\}$ in C_r and $\tilde{B}$ in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$, let σ_1 be an $(r-1)$-subspace in $T(p_1)$ containing no point of $B(T(p_1))$. The collineation ω_{r,p_1} maps σ_1 onto a C_{r-1} in C_r containing no point of B. Moreover, the dimension of the subspace τ in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$ spanned by C_{r-1} is $\frac{1}{2}(r-1)(r+2)$ and τ is skew with $T(p_1)$; otherwise, $T(p_1) \cup \tau$ spans a hyperplane in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$ containing C_r. Therefore, $\sigma_1 \cup \tau$ spans a hyperplane α. Since $\omega_{p_1p_i} = \omega_{r,p_i}^{-1} \circ \omega_{r,p_1}$ and the tangent spaces to C_r at the points of C_{r-1} belong to α, the $(r-1)$-subspaces $\sigma_j = \omega_{p_1p_i}(\sigma_1)$ belong to α. Thus, the next proposition holds.

Proposition 4.9. *Let $B = \{p_1, \ldots, p_{r+1}\}$ be a basis on C_r and σ_i a $(r-1)$-subspace in the tangent space $T(p_i)$ containing no point of $B(T(p_i))$. If $\omega_{p_ip_j}(\sigma_i) = \sigma_j$, then the $(r-1)$-subspaces $\sigma_1, \ldots, \sigma_{r+1}$ belong to a hyperplane in* $\mathrm{PG}(\frac{1}{2}r(r+3), q)$.

Next, let C_r and $V(r, q)$ be the Veronese cap and the Veronese variety in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$, respectively. $S(r, q)$ denoting the set of all conics in $V(r, q)$, $(V(r, q), S(r, q))$ is a Veronese cap of index r (see Remark 1). Denote by $B = \{p_1, \ldots, p_{r+1}\}$ and $B' = \{q_1, \ldots, q_{r+1}\}$ two bases of C_r and $V(r, q)$, respectively, and $\tilde{B}$, $\tilde{B}'$ the bases of $\mathrm{PG}(\frac{1}{2}r(r+3), q)$ associated with B and B' (see Proposition 4.8). Moreover, let σ_i and σ_i' be two $(r-1)$-subspaces in the tangent spaces $T(p_i)$ and $T(q_i)$ to C_r and $V(r, q)$, respectively, verifying Proposition 4.9 and α, α' the hyperplanes in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$ containing σ_i and σ_i', respectively. By the same argument in [5, n.5 and n.6] the projectivity Ω in $\mathrm{PG}(\frac{1}{2}r(r+3), q)$, mapping p_i onto q_i, $p_{ij} = T(p_i) \cap T(p_j)$ onto $q_{ij} = T(q_i) \cap T(q_j)$ and α onto α', maps the tangent spaces to C_r onto the tangent spaces to $V(r, q)$, since $\Omega \circ \omega_{p_ip_j} = \omega_{q_iq_j} \circ \Omega$. Whence $\Omega(C_r) = V(r, q)$ and the next result follows.

Theorem 2. *Let $C_r = (C, S)$ be a Veronese cap of index r in* $\mathrm{PG}(n, q)$. *Then*

(i) $n = \frac{1}{2}r(r+3)$.

(ii) *C_r is a Veronese variety $V(r, q)$.*

(iii) *S is the set of all conics contained in C_r.*

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